Identification of covariance structures

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Covariance structures

Definition of a *covariance structure*: parametric model featuring A and B such that

$$A(\theta)\Sigma A(\theta)' = B(\theta)B(\theta)'$$

$$\begin{aligned} \Sigma &= V(y_t|\mathscr{F}_t) \\ y_t &: \text{ observable variables} \\ \theta &: \text{ parameters} \end{aligned}$$

Examples:

- Structural VARs
- Factor Multivariate GARCH/SV models
- Multivariate statistics (factor analysis, principal components, . . .)
- LISREL models

The identification problem

If the equation

$$A_0 \Sigma A_0' = B_0 B_0'$$

is satisfied, then

$$A_1 \Sigma A_1' = B_1 B_1'$$

is satisfied too, with

$$\begin{aligned} A_1 &= Q A_0 \\ B_1 &= Q B_0 H \end{aligned}$$

where Q is invertible and H is orthogonal.

What constraints are needed on A and B to ensure that

$$\Sigma = A^{-1}BB'(A')^{-1}.$$

is unique?
In general, it never is `:-)`.
In an ϵ-neighbourhood, impose *linear constraints*:

$$\begin{aligned} R_a \text{vec} A = R_a a &= d_a \\ R_b \text{vec} B = R_b b &= d_b; \end{aligned}$$

or, in explicit form,

$$\begin{aligned} a &= S_a \theta + d_a \\ b &= S_b \theta + d_b; \end{aligned}$$

In vectorised form: if

$$\begin{aligned} A + \mathrm{d}A &= (I+Q)A \\ B + \mathrm{d}B &= (I+Q)B(I+H), \end{aligned}$$

where $(I+Q)$ is nonsingular and $(I+H)$ is orthogonal, then

$$\begin{aligned} \mathrm{d}a &= (I \otimes Q)(S_a\theta + s_a) \\ \mathrm{d}b &= \left[(I \otimes Q) + (H' \otimes I)\right](S_b\theta + s_b) \end{aligned}$$

Identification fails if H and/or Q exist such that

$$\begin{aligned} R_a' \mathrm{d}a = R_a'(I \otimes Q)(S_a\theta + s_a) &= 0 \\ R_b' \mathrm{d}b = R_b' \left[(I \otimes Q) + (H' \otimes I) \right] (S_b\theta + s_b) &= 0 \end{aligned}$$

What can we say on H and Q?

Let's start from a narrower case.

The C-model

$$\hat{\Sigma} = B_0 B_0'.$$

The model is under-identified if an admissible B_1 exists in a neighbourhood of B_0 which can be written as

$$B_1 = B_0 + \mathrm{d}B_0 = B_0(I + H),$$

where $(I + H)$ is orthogonal.
Then $(I + H)$ must be an *infinitesimal rotation*.

Infinitesimal rotations

A matrix $I + H$ is an infinitesimal rotation if

- H is infinitesimal
- $(I + H)$ is orthogonal.

This implies that H is hemisymmetric ($H = -H'$) and

$$\underset{n^2\times 1}{h} = \text{vec}H = \underset{n^2\times\frac{n(n-1)}{2}}{\widetilde{D}_n} \cdot \underset{\frac{n(n-1)}{2}\times 1}{\varphi}$$

Infinitesimal rotations — example

For example, a (2×2) infinitesimal rotation can be written as

$$I + H = \begin{bmatrix} 1 & -\varphi \\ \varphi & 1 \end{bmatrix}$$

So that

$$h = \begin{bmatrix} 0 \\ \varphi \\ -\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \varphi$$

The C-model — continued

If

$$R_b' \mathrm{d}b = R_b'(H' \otimes I)(S_b\theta + s_b) = 0.$$

where H is hemisymmetric $\Longrightarrow$ no identification at θ. It can be shown that if the model is identified at θ, the equations

(1) $$\varphi' \widetilde{D}_n'(I \otimes R_i')[S\theta + s] = \varphi' T_i\theta + \varphi' t_i = 0$$

have no solutions except $\varphi = 0$ for $i = 1, \ldots, p$.

The C-model — continued

Various cases:

- If the order condition fails, there is a $\varphi \neq 0$ that satisfies (1) for any θ.
- The order condition is met, but a $\varphi \neq 0$ can be found that satisfies (1) for any θ. This case is called *structural under-identification*.

The structure condition can be checked via the rank of

$$\mathcal{M} = \sum_{i=1}^{n^2-p} \left[\widetilde{D}_n'(I \otimes S_i')RR'(I \otimes S_i)\widetilde{D}_n \right] + \widetilde{D}_n'(I \otimes \bar{S}')RR'(I \otimes \bar{S})\widetilde{D}_n$$

Main results

- Both the order and structure conditions are necessary. Neither is sufficient. However,
- If both the order and structure conditions are met, then identification occurs *almost everywhere* in Θ.
- Both conditions can be checked numerically.

The C-model — example

The classic Cholesky decomposition:

$$B = \begin{bmatrix} \theta_1 & 0 \\ \theta_2 & \theta_3 \end{bmatrix}$$

- Order condition: OK
- $\mathcal{M} = \widetilde{D}_n' \left[\sum_{i=1}^{3} (I \otimes S_i) R' R (I \otimes S_i')\right] \widetilde{D}_n = 1$
- therefore the structure condition is OK too.

The C-model — example

Analytically:
Consider postmultiplying B by an arbitrary infinitesimal rotation:

$$\begin{bmatrix} \theta_1 & 0 \\ \theta_2 & \theta_3 \end{bmatrix} \begin{bmatrix} 1 & -\varphi \\ \varphi & 1 \end{bmatrix} = \begin{bmatrix} \theta_1 & -\theta_1\varphi \\ \theta_2 + \theta_3\varphi & \theta_3 - \theta_2\varphi \end{bmatrix}$$

The result is not admissible (ie lower triangular) if and only if $\theta_1\varphi = 0$. If $\varphi \neq 0$, then θ_1 must be 0.

The C-model — example

In terms of the previous results:
equation (1) is

$$\varphi' \widetilde{D}_n'(I \otimes R_i')(S\theta + s) = \varphi \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \varphi \cdot \theta_1 = 0.$$

If $\varphi \neq 0$, then the only solution is $\theta_1 = 0$.
The only region in Θ where the model is under-identified is the plane $\theta_1 = 0$, which has zero Lebesgue measure in $\mathbb{R}^3$.
As a consequence, the model is identified almost everywhere.

The AB-model

In the general case, we have

$$\begin{aligned} A + \mathrm{d}A &= (I+Q)A \\ B + \mathrm{d}B &= (I+Q)B(I+H). \end{aligned}$$

Under-identification implies non-zero solutions to

$$\begin{aligned} R_a' \mathrm{d}a = R_a'(I \otimes Q)(S_a\theta + s_a) &= 0 \\ R_b' \mathrm{d}b = R_b' \left[(I \otimes Q) + (H' \otimes I)\right](S_b\theta + s_b) &= 0; \end{aligned}$$

The AB-model — continued

H must be hemisymmetric (like before). Q is unrestricted because $(I+Q)$ is nonsingular for any infinitesimal Q. Again, identification implies no trivial solutions to

$$\begin{aligned} q'U_i^a\theta + q'u_i^a &= 0 \quad \text{for } i = 1 \ldots p_a \\ q'U_j^b\theta + \varphi'T_j^b\theta + q'u_j^b + \varphi't_j^b &= 0 \quad \text{for } j = 1 \ldots p_b \end{aligned}$$

Again, the matrices U_i^a, U_i^b and T_i^b depend only on the constraints structure.

The AB-model — structure condition

The AB-model is structurally identified if a certain matrix $\mathcal{M}$ (that does not fit in a slide) is invertible.
$\mathcal{M}$ is a function of the R's and S's alone.

$$\Downarrow$$

The structure condition can, again, be checked numerically.

Open questions

- Relationship between the set of non-identified points and the set of singular matrices (both have 0 measure)?
- Can this be generalised to nonlinear constraints?
- How about inequality constraints?